CHAPTER 7

Term-II

INTEGRALS

Syllabus

- ***Integration as inverse process of differentiation. Integration of a variety of functions by substitution, by partial fractions and by parts. Evaluation of simple integrals of the following types and problems based on them.***

$$\int \frac{dx}{x^2 \pm a^2}, \int \frac{dx}{\sqrt{x^2 \pm a^2}}, \int \frac{dx}{\sqrt{a^2 - x^2}}, \int \frac{dx}{ax^2+bx+c}, \int \frac{dx}{\sqrt{ax^2+bx+c}}, \int \frac{px+q}{ax^2+bx+c}dx,$$

$$\int \frac{px+q}{\sqrt{ax^2+bx+c}}dx, \int \sqrt{a^2 \pm x^2}dx, \int \sqrt{x^2-a^2}dx$$

Fundamental Theorem of Calculus (without proof). Basic properties of definite integrals and evaluation of definite integrals.

STAND ALONE MCQs (1 Mark each)

Q. 1. $\int \frac{\cos 2x - \cos 2\theta}{\cos x - \cos\theta} dx$ is equal to

(A) $2(\sin x + x\cos\theta) + C$

(B) $2(\sin x - x\cos\theta) + C$

(C) $2(\sin x + 2x\cos\theta) + C$

(D) $2(\sin x - 2x\cos\theta) + C$

Ans. Option (A) is correct.

Explanation : Let,

$$I = \int \frac{\cos 2x - \cos 2\theta}{\cos x - \cos\theta} dx$$

$$= \int \frac{(2\cos^2 x - 1 - 2\cos^2\theta + 1)}{\cos x - \cos\theta} dx$$

$$= 2\int \frac{(\cos x + \cos\theta)(\cos x - \cos\theta)}{(\cos x - \cos\theta)} dx$$

$$= 2\int (\cos x + \cos\theta) dx$$

$$= 2\sin x + 2x\cos\theta + C$$

Q. 2. The value of $\int_{-\pi/2}^{\pi/2} (x^3 + x\cos x + \tan^5 x + 1)\, dx$ is

(A) 0 **(B)** 2

(C) π **(D)** 1

Ans. Option (C) is correct.

Explanation : Let,

$$I = \int_{-\pi/2}^{\pi/2} (x^3 + x\cos x + \tan^5 x + 1)\, dx$$

$$= \int_{-\pi/2}^{\pi/2} x^3 dx + \int_{-\pi/2}^{\pi/2} x\cos x + \int_{-\pi/2}^{\pi/2} \tan^5 x dx + \int_{-\pi/2}^{\pi/2} 1 \cdot dx$$

It is known that if $f(x)$ is an even function, then

$$\int_{-a}^{a} f(x)dx = 2\int_0^a f(x)dx$$

and if $f(x)$ is an odd function, then

$$\int_{-a}^{a} f(x)dx = 0$$

$$\therefore\ I = 0 + 0 + 0 + 2\int_0^{\pi/2} 1 \cdot dx$$

$$= 2[x]_0^{\pi/2} = \frac{2\pi}{2} = \pi$$

Q. 3. $\int \frac{dx}{e^x + e^{-x}}$ is equal to

(A) $\tan^{-1}(e^x) + C$
(B) $\tan^{-1}(e^{-x}) + C$
(C) $\log(e^x - e^{-x}) + C$
(D) $\log(e^x + e^{-x}) + C$

Ans. Option (A) is correct.

Explanation:

Let $I = \int \frac{dx}{e^x + e^{-x}} dx$

$= \int \frac{e^x}{e^{2x}+1} dx$

Also, let $e^x = t$

$\Rightarrow e^x dx = dt$

$\Rightarrow I = \int \frac{dt}{1+t^2}$

$= \tan^{-1} t + C$

$= \tan^{-1}(e^x) + C$

Q. 4. $\int \frac{\cos 2x}{(\sin x + \cos x)^2} dx$ is equal to

(A) $\frac{-1}{\sin x + \cos x} + C$
(B) $\log|\sin x + \cos x| + C$
(C) $\log|\sin x - \cos x| + C$
(D) $\frac{1}{(\sin x + \cos x)^2}$

Ans. Option (B) is correct.

Explanation:

Let $I = \frac{\cos 2x}{(\cos x + \sin x)^2}$

$I = \int \frac{\cos^2 x - \sin^2 x}{(\cos x + \sin x)^2} dx$

$= \int \frac{(\cos x + \sin x)(\cos x - \sin x)}{(\cos x + \sin x)^2} dx$

$= \int \frac{\cos x - \sin x}{\cos x + \sin x} dx$

Let $\cos x + \sin x = t$

$\Rightarrow (\cos x - \sin x)\, dx = dt$

$\Rightarrow I = \int \frac{dt}{t}$

$= \log|t| + C$

$= \log|\cos x + \sin x| + C$

Q. 5. If $f(a+b-x) = f(x)$, then $\int_a^b x f(x) dx$ is equal to

(A) $\frac{a+b}{2}\int_a^b f(b-x)dx$ (B) $\frac{a+b}{2}\int_a^b f(b+x)dx$

(C) $\frac{b-a}{2}\int_a^b f(x)dx$ (D) $\frac{a+b}{2}\int_a^b f(x)dx$

Ans. Option (D) is correct.

Explanation:

Let $I = \int_a^b x f(x) dx$ ...(i)

$I = \int_a^b (a+b-x) f(a+b-x) dx$

$\left[\because \int_a^b f(x)dx = \int_a^b f(a+b-x)dx\right]$

$\Rightarrow I = \int_a^b (a+b-x) f(x) dx$

$\Rightarrow I = (a+b)\int_a^b f(x)dx - I$ [Using Eq. (i)]

$\Rightarrow I + I = (a+b)\int_a^b f(x)dx$

$\Rightarrow 2I = (a+b)\int_a^b f(x)dx$

$\Rightarrow I = \left(\frac{a+b}{2}\right)\int_a^b f(x)dx$

Q. 6. The value of $\int_0^1 \tan^{-1}\left(\frac{2x-1}{1+x-x^2}\right) dx$ is

(A) 1 (B) 0

(C) –1 (D) $\frac{\pi}{4}$

Ans. Option (B) is correct.

Explanation :

Let $I = \int_0^1 \tan^{-1}\left(\frac{2x-1}{1+x-x^2}\right)dx$

$\Rightarrow I = \int_0^1 \tan^{-1}\left(\frac{x-(1-x)}{1+x(1-x)}\right)dx$

$\Rightarrow I = \int_0^1 [\tan^{-1} x - \tan^{-1}(1-x)]dx$ ...(i)

$\Rightarrow I = \int_0^1 [\tan^{-1}(1-x) - \tan^{-1}(1-1+x)]dx$

$\Rightarrow I = \int_0^1 [\tan^{-1}(1-x) - \tan^{-1}(x)]dx$

$\Rightarrow I = \int_0^1 [\tan^{-1}(1-x) - \tan^{-1}(x)]dx$ ...(ii)

Adding equations (i) and (ii), we obtain

$2I = \int_0^1 (\tan^{-1} x + \tan^{-1}(1-x)$

$- \tan^{-1}(1-x) - \tan^{-1} x)dx$

$\Rightarrow 2I = 0$

$\Rightarrow I = 0$

Q. 7. $\frac{dx}{\sin(x-a)\,\sin(x-b)}$ is equal to

(A) $\sin(b-a)\log\left|\frac{\sin(x-b)}{\sin(x-a)}\right| + C$

(B) $\operatorname{cosec}(b-a)\log\left|\frac{\sin(x-a)}{\sin(x-b)}\right| + C$

(C) $\operatorname{cosec}(b-a)\log\left|\frac{\sin(x-b)}{\sin(x-a)}\right| + C$

(D) $\sin(b-a)\log\left|\frac{\sin(x-a)}{\sin(x-b)}\right| + C$

Ans. Option (C) is correct.

Explanation : Let,

$$I=\int\frac{dx}{\sin(x-a)\sin(x-b)}$$

$$=\frac{1}{\sin(b-a)}\int\frac{\sin(b-a)}{\sin(x-a)\sin(x-b)}dx$$

$$=\frac{1}{\sin(b-a)}\int\frac{\sin(x-a-x+b)}{\sin(x-a)\sin(x-b)}dx$$

$$=\frac{1}{\sin(b-a)}\int\frac{\sin\{(x-a)-(x-b)\}}{\sin(x-a)\sin(x-b)}dx$$

$$=\frac{1}{\sin(b-a)}\int\frac{\sin(x-a)\cos(x-b)-\cos(x-a)\sin(x-b)}{\sin(x-a)\sin(x-b)}dx$$

$$=\frac{1}{\sin(b-a)}\int[\cot(x-b)-\cot(x-a)]dx$$

$$=\frac{1}{\sin(b-a)}[\log|\sin(x-b)|-\log|\sin(x-a)|]+C$$

$$=\operatorname{cosec}(b-a)\log\left|\frac{\sin(x-b)}{\sin(x-a)}\right|+C$$

Q. 8. $\int\sqrt{1+x^2}\,dx$ is equal to

(A) $\frac{x}{2}\sqrt{1+x^2}+\frac{1}{2}\log\left|\left(x+\sqrt{1+x^2}\right)\right|+C$

(B) $\frac{2}{3}(1+x^2)^{3/2}+C$

(C) $\frac{2}{3}x(1+x^2)^{3/2}+C$

(D) $\frac{x^2}{2}\sqrt{1+x^2}+\frac{1}{2}x^2\log\left(x+\sqrt{1+x^2}\right)+C$

Ans. Option (A) is correct.

Explanation : It is known that,

$$\int\sqrt{a^2+x^2}\,dx=\frac{x}{2}\sqrt{a^2+x^2}+\frac{a^2}{2}\log\left|x+\sqrt{x^2+a^2}\right|+C$$

$$\therefore\quad\int\sqrt{1+x^2}\,dx=\frac{x}{2}\sqrt{1+x^2}+\frac{1}{2}\log\left|x+\sqrt{1+x^2}\right|+C$$

Q. 9. $\int\frac{x\,dx}{(x-1)(x-2)}$ equals

(A) $\log\left|\frac{(x-1)^2}{x-2}\right|+C$

(B) $\log\left|\frac{(x-2)^2}{x-1}\right|+C$

(C) $\log\left|\left(\frac{x-1}{x-2}\right)^2\right|+C$

(D) $\log|(x-1)(x-2)|+C$

Ans. Option (B) is correct.

Explanation :

Let $\frac{x}{(x-1)(x-2)}=\frac{A}{(x-1)}+\frac{B}{(x-2)}$

$x=A(x-2)+B(x-1)$ ...(i)

Substituting $x = 1$ and 2 in Eq. (i), we obtain

A = –1 and B = 2

$$\therefore\quad\frac{x}{(x-1)(x-2)}=-\frac{1}{(x-1)}+\frac{2}{(x-2)}$$

$$\Rightarrow\int\frac{x}{(x-1)(x-2)}dx=\int\left\{\frac{-1}{(x-1)}+\frac{2}{(x-2)}\right\}dx$$

$$=-\log|x-1|+2\log|x-2|+C$$

$$=\log\left|\frac{(x-2)^2}{x-1}\right|+C$$

Q. 10. If $f(x)=\int_0^x t\sin t\,dt$, then $f'(x)$ is

(A) $\cos x + x\sin x$

(B) $x\sin x$

(C) $x\cos x$

(D) $\sin x + x\cos x$

Ans. Option (B) is correct.

Explanation :

$$f(x)=\int_0^x t\sin t\,dt$$

Integrating by parts, we obtain

$$f(x)=t\int_0^x\sin t\,dt-\int_0^x\left\{\left(\frac{d}{dt}t\right)\int\sin t\,dt\right\}dt$$

$$=[t(-\cos t)]_0^x-\int_0^x(-\cos t)\,dt$$

$$=[-t\cos t+\sin t]_0^x$$

$$=-x\cos x+\sin x$$

$$\Rightarrow\quad f'(x)=-[\{x(-\sin x)\}+\cos x]+\cos x$$

$$=x\sin x-\cos x+\cos x$$

$$=x\sin x$$

Q. 11. $\int\tan^{-1}\sqrt{x}\,dx$ is equal to

(A) $(x+1)\tan^{-1}\sqrt{x}-\sqrt{x}+C$

(B) $x\tan^{-1}\sqrt{x}-\sqrt{x}+C$

(C) $\sqrt{x}-x\tan^{-1}\sqrt{x}+C$

(D) $\sqrt{x}-(x+1)\tan^{-1}\sqrt{x}+C$

Ans. Option (A) is correct.

Explanation : Let,

$$I=\int 1\cdot\tan^{-1}\sqrt{x}\,dx$$

$$=\tan^{-1}\sqrt{x}\cdot x-\frac{1}{2}\int\frac{1}{(1+x)}\cdot\frac{2}{\sqrt{x}}dx$$

$$=x\tan^{-1}\sqrt{x}-\frac{1}{2}\int\frac{2}{\sqrt{x}(1+x)}dx$$

Put $x = t^2$

$\Rightarrow dx = 2t\,dt$

$\therefore \quad I = x\tan^{-1}\sqrt{x} - \int \frac{t}{t(1+t^2)}dt$

$= x\tan^{-1}\sqrt{x} - \int \frac{t^2}{1+t^2}dt$

$= x\tan^{-1}\sqrt{x} - \int \left(1 - \frac{1}{1+t^2}\right)dt$

$= x\tan^{-1}\sqrt{x} - \sqrt{x} + \tan^{-1}t + C$

$= x\tan^{-1}\sqrt{x} - \sqrt{x} + \tan^{-1}\sqrt{x} + C$

$- (x+1)\tan^{-1}\sqrt{x} - \sqrt{x} + C$

Q. 12. $\int x^2 e^{x^3}dx$ is equal to

(A) $\frac{1}{3}e^{x^3}+C$ (B) $\frac{1}{3}e^{x^2}+C$

(C) $\frac{1}{2}e^{x^3}+C$ (D) $\frac{1}{2}e^{x^2}+C$

Ans. Option (A) is correct.

Explanation:

Let $I = \int x^2 e^{x^3}dx$

Also, let $x^3 = t$

$\Rightarrow 3x^2dx = dt$

$\Rightarrow I = \frac{1}{3}\int e^t dt$

$= \frac{1}{3}(e^t)+C$

$= \frac{1}{3}e^{x^3}+C$

Q. 13. $\int e^x \sec x(1+\tan x)dx$ is equal to

(A) $e^x \cos x + C$ (B) $e^x \sec x + C$

(C) $e^x \sin x + C$ (D) $e^x \tan x + C$

Ans. Option (B) is correct.

Explanation : $\int e^x \sec x(1+\tan x)dx$

Let

$I = \int e^x \sec x(1+\tan x)dx = \int e^x(\sec x + \sec x\tan x)dx$

Also, let $\sec x = f(x) \Rightarrow \sec x\tan x = f'(x)$

It is known that, $\int e^x\{f(x)+f'(x)\}dx = e^x f(x)+C$

$\therefore I = e^x \sec x + C$

Q. 14. $\int \frac{\sin^2 x - \cos^2 x}{\sin^2 x\ \cos^2 x}dx$ is equal to

(A) $\tan x + \cot x + C$

(B) $\tan x + \text{cosec}\, x + C$

(C) $-\tan x + \cot x + C$

(D) $\tan x + \sec x + C$

Ans. Option (A) is correct.

Explanation:

$\int \frac{\sin^2 x - \cos^2 x}{\sin^2 x\cos^2 x}dx = \int \left(\frac{\sin^2 x}{\sin^2 x\cos^2 x} - \frac{\cos^2 x}{\sin^2 x\cos^2 x}\right)dx$

$= \int(\sec^2 x - \text{cosec}^2 x)dx$

$= \tan x + \cot x + C$

Q. 15. $\int \frac{dx}{\sin^2 x\cos^2 x}$ is equal to

(A) $\tan x + \cot x + C$ (B) $\tan x - \cot x + C$

(C) $\tan x\cot x + C$ (D) $\tan x - \cot 2x + C$

Ans. Option (B) is correct.

Explanation:

Let $I = \int \frac{dx}{\sin^2 x\cos^2 x}$

$= \int \frac{1}{\sin^2 x\cos^2 x}dx$

$= \int \frac{\sin^2 x + \cos^2 x}{\sin^2 x\cos^2 x}dx$

$= \int \frac{\sin^2 x}{\sin^2 x\cos^2 x}dx + \int \frac{\cos^2 x}{\sin^2 x\cos^2 x}dx$

$= \int \sec^2 x\,dx + \int \text{cosec}^2\ x\,dx$

$= \tan x - \cot x + C$

Q. 16. $\int \frac{x^9}{(4x^2+1)^6}dx$ is equal to

(A) $\frac{1}{5x}\left(4+\frac{1}{x^2}\right)^{-5}+C$ (B) $\frac{1}{5}\left(4+\frac{1}{x^2}\right)^{-5}+C$

(C) $\frac{1}{10x}(1+4)^{-5}+C$ (D) $\frac{1}{10}\left(\frac{1}{x^2}+4\right)^{-5}+C$

Ans. Option (D) is correct.

Explanation : Let

$I = \int \frac{x^9}{(4x^2+1)^6}dx$

$= \int \frac{x^9}{x^{12}\left(4+\frac{1}{x^2}\right)^6}dx$

$= \int \frac{dx}{x^3\left(4+\frac{1}{x^2}\right)^6}$

Put $4+\frac{1}{x^2} = t$

$\Rightarrow \frac{-2}{x^3}dx = dt$

$\Rightarrow \frac{1}{x^3}dx = -\frac{1}{2}dt$

$\therefore \quad I = -\frac{1}{2}\int \frac{dt}{t^6}$

$= -\frac{1}{2}\left[\frac{t^{-6+1}}{-6+1}\right]+C$

$$= \frac{1}{10}\left[\frac{1}{t^5}\right] + C$$

$$= \frac{1}{10}\left(4 + \frac{1}{x^2}\right)^{-5} + C$$

Q. 17. $\int \frac{x^3}{x+1}$ is equal to

(A) $x + \frac{x^2}{2} + \frac{x^3}{3} - \log|1-x| + C$

(B) $x + \frac{x^2}{2} - \frac{x^3}{3} - \log|1-x| + C$

(C) $x - \frac{x^2}{2} - \frac{x^3}{3} - \log|1+x| + C$

(D) $x - \frac{x^2}{2} + \frac{x^3}{3} - \log|1+x| + C$

Ans. Option (D) is correct.

Explanation : Let,

$$I = \int \frac{x^3}{x+1} dx$$

$$= \int \left((x^2 - x + 1) - \frac{1}{(x+1)}\right) dx$$

$$= \frac{x^3}{3} - \frac{x^2}{2} + x - \log|x+1| + C$$

Q. 18. If $\int \frac{x^3 dx}{\sqrt{1+x^2}} = a(1+x^2)^{3/2} + b\sqrt{1+x^2} + C$, then

(A) $a = \frac{1}{3}, \ b = 1$ (B) $a = \frac{-1}{3}, \ b = 1$

(C) $a = \frac{-1}{3}, \ b = -1$ (D) $a = \frac{1}{3}, \ b = -1$

Ans. Option (D) is correct.

Explanation: Let,

$$I = \int \frac{x^3}{\sqrt{1+x^2}} dx$$

$$= a(1+x^2)^{3/2} + b\sqrt{1+x^2} + C$$

$\therefore$ $$I = \int \frac{x^3}{\sqrt{1+x^2}} dx$$

$$= \int \frac{x^2 \cdot x}{\sqrt{1+x^2}} dx$$

Put $1 + x^2 = t^2$

$\Rightarrow$ $2xdx = 2tdt$

$\therefore$ $$I = \int \frac{t(t^2-1)}{t} dt$$

$$= \frac{t^3}{3} - t + C$$

$$= \frac{1}{3}(1+x^2)^{3/2} - \sqrt{1+x^2} + C$$

$\therefore$ $a = \frac{1}{3}$ and $b = -1$

Q. 19. $\int_{-\pi/4}^{\pi/4} \frac{dx}{1+\cos 2x}$ is equal to

(A) 1 (B) 2

(C) 3 (D) 4

Ans. Option (A) is correct.

Explanation : Let

$$I = \int_{-\pi/4}^{\pi/4} \frac{dx}{1+\cos 2x}$$

$$= \int_{-\pi/4}^{\pi/4} \frac{dx}{2\cos^2 x}$$

$$= \frac{1}{2}\int_{-\pi/4}^{\pi/4} \sec^2 x dx$$

$$= \int_0^{\pi/4} \sec^2 x dx$$

$$= [\tan x]_0^{\pi/4}$$

$$= 1$$

Q. 20. $\int \frac{dx}{x^2 + 2x + 2}$ equals

(A) $x\tan^{-1}(x+1) + C$ (B) $\tan^{-1}(x+1) + C$

(C) $(x+1)\, x\tan^{-1} + C$ (D) $\tan^{-1} + C$

Ans. Option (B) is correct.

Explanation :

$$\int \frac{dx}{x^2+2x+2} = \int \frac{dx}{(x^2+2x+1)+1}$$

$$= \int \frac{1}{(x+1)^2 + (1)^2} dx$$

$$= \left[\tan^{-1}(x+1)\right] + C$$

Q. 21. $\int_0^{\pi/2} \sqrt{1 - \sin 2x}dx$ is equal to

(A) $2\sqrt{2}$ (B) $2(\sqrt{2}+1)$

(C) 2 (D) $2(\sqrt{2}-1)$

Ans. Option (D) is correct.

Explanation: Let

$$I = \int_0^{\pi/2} \sqrt{1-\sin 2x}dx$$

$$= \int_0^{\pi/4} \sqrt{(\cos x - \sin x)^2}\, dx + \int_{\pi/4}^{\pi/2} \sqrt{(\sin x - \cos x)^2}\, dx$$

$$= [\sin x + \cos x]_0^{\pi/4} + [-\cos x - \sin x]_{\pi/4}^{\pi/2}$$

$$= \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} - 0 - 1 + \left(-0 - 1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}\right)$$

$$= 2\sqrt{2} - 2$$

$$= 2(\sqrt{2} - 1)$$

Q. 22. The anti-derivative of $\left(\sqrt{x} + \frac{1}{\sqrt{x}}\right)$ equals

(A) $\frac{1}{3}x^{1/3} + 2x^{1/2} + C$ (B) $\frac{2}{3}x^{2/3} + \frac{1}{2}x^2 + C$

(C) $\frac{2}{3}x^{2/3} + 2x^{1/2} + C$ (D) $\frac{3}{2}x^{3/2} + \frac{1}{2}x^{1/2} + C$

Ans. Option (C) is correct.

Explanation:

$$\left(\sqrt{x}+\frac{1}{\sqrt{x}}\right)dx = \int x^{1/2}dx + \int x^{-1/2}dx$$

$$= \frac{x^{3/2}}{\frac{3}{2}} + \frac{x^{1/2}}{\frac{1}{2}} + C$$

$$= \frac{2}{3}x^{3/2} + 2x^{1/2} + C$$

Q. 23. $\int_1^{\sqrt{3}} \frac{dx}{1+x^2}$ equals

(A) $\frac{\pi}{3}$ (B) $\frac{2\pi}{3}$

(C) $\frac{\pi}{6}$ (D) $\frac{\pi}{12}$

Ans. Option (D) is correct.

Explanation:

$$\int \frac{dx}{1+x^2} = \tan^{-1} x = F(x)$$

By second fundamental theorem of calculus, we obtain

$$\int_1^{\sqrt{3}} \frac{dx}{1+x^2} = F\left(\sqrt{3}\right) - F(1)$$

$$- \tan^{-1}\sqrt{3} - \tan^{-1}1$$

$$= \frac{\pi}{3} - \frac{\pi}{4}$$

$$= \frac{\pi}{12}$$

Q. 24. $\int_0^{2/3} \frac{dx}{4+9x^2}$ equals

(A) $\frac{\pi}{6}$ (B) $\frac{\pi}{12}$

(C) $\frac{\pi}{24}$ (D) $\frac{\pi}{4}$

Ans. Option (C) is correct.

Explanation :

$$\int \frac{dx}{4+9x^2} = \int \frac{dx}{(2)^2+(3x)^2}$$

Put $3x = t$

$\Rightarrow 3dx = dt$

$$\therefore \int \frac{dx}{(2)^2+(3x)^2} = \frac{1}{3}\int \frac{dt}{(2)^2+(t)^2}$$

$$= \frac{1}{3}\left[\frac{1}{2}\tan^{-1}\frac{t}{2}\right]$$

$$= \frac{1}{6}\tan^{-1}\left(\frac{3x}{2}\right)$$

$$= F(x)$$

By second fundamental theorem of calculus, we obtain

$$\int_0^{2/3} \frac{dx}{4+9x^2} = F\left(\frac{2}{3}\right) - F(0)$$

$$= \frac{1}{6}\tan^{-1}\left[\frac{3}{2}\times\frac{2}{3}\right] - \frac{1}{6}\tan^{-1}(0)$$

$$= \frac{1}{6}\tan^{-1}(1)$$

$$= \frac{1}{6}\tan^{-1}\left[\tan\frac{\pi}{4}\right]$$

$$= \frac{\pi}{24}$$

Q. 25. If $\frac{d}{dx}f(x) = 4x^3 - \frac{3}{x^4}$ such that $f(2) = 0$. Then $f(x)$ is

(A) $x^4 + \frac{1}{x^3} - \frac{129}{8}$ (B) $x^3 + \frac{1}{x^4} + \frac{129}{8}$

(C) $x^4 + \frac{1}{x^3} + \frac{129}{8}$ (D) $x^3 + \frac{1}{x^4} - \frac{129}{8}$

Ans. Option (A) is correct.

Explanation : It is given that,

$$\frac{d}{dx}f(x) = 4x^3 - \frac{3}{x^4}$$

$\therefore$ Anti-derivative of $4x^3 - \frac{3}{x^4} = f(x)$

$$\therefore \quad f(x) = \int 4x^3 - \frac{3}{x^4}dx$$

$$f(x) = 4\int x^3dx - 3\int\left(x^{-4}\right)dx$$

$$\therefore \quad f(x) = 4\left(\frac{x^4}{4}\right) - 3\left(\frac{x^{-3}}{-3}\right) + C$$

$$f(x) = x^4 + \frac{1}{x^3} + C$$

Also,

$$f(2) = 0$$

$$\therefore \quad f(2) = (2)^4 + \frac{1}{(2)^3} + C$$

$$= 0$$

$$\Rightarrow \quad 16 + \frac{1}{8} + C = 0$$

$$\Rightarrow \quad C = -\left(16 + \frac{1}{8}\right)$$

$$\Rightarrow \quad C = \frac{-129}{8}$$

$$\therefore \quad f(x) = x^4 + \frac{1}{x^3} - \frac{129}{8}$$

ASSERTION AND REASON BASED MCQs (1 Mark each)

Directions : In the following questions, A statement of Assertion (A) is followed by a statement of Reason (R). Mark the correct choice as:

(A) Both A and R are true and R is the correct explanation of A

(B) Both A and R are true but R is NOT the correct explanation of A

(C) A is true but R is false

(D) A is false and R is True

Q. 1. Assertion (A): $\int \frac{dx}{x^2+2x+3} = \frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{x+1}{\sqrt{2}}\right)+c$

Reason (R): $\int \frac{dx}{x^2+a^2} = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right)+c$

Ans. Option (A) is correct.

Explanation:

$$\int \frac{dx}{x^2+a^2} = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right)+c.$$

This is a standard integral and hence R is true.

$$\int \frac{dx}{x^2+2x+3} = \int \frac{dx}{(x+1)^2+(\sqrt{2})^2}$$
$$= \frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{x+1}{\sqrt{2}}\right)+c$$

Hence A is true and R is the correct explanation for A.

Q. 2. Assertion(A): $\int e^x[\sin x - \cos x]\,dx = e^x \sin x + C$

Reason (R): $\int e^x[f(x)+f'(x)]\,dx = e^x f(x)+c$

Ans. Option (D) is correct.

Explanation:

$$\int e^x[f(x)+f'(x)]dx = \int e^x f(x)dx + \int e^x f'(x)dx$$
$$= f(x)e^x - \int f'(x)e^x dx + \int f'(x)e^x dx$$
$$= e^x f(x)+c$$

Hence R is true.

$$\int e^x(\sin x - \cos x)dx = e^x(-\cos x)+c$$
$$= -e^x \cos x + c$$
$$\left[\because \frac{d}{dx}(-\cos x) = \sin x\right]$$

Hence A is false.

Q. 3. Assertion (A): $\int x^x(1+\log x)dx = x^x + c$

Reason (R): $\frac{d}{dx}(x^x) = x^x(1+\log x)$

Ans. Option (A) is correct.

Explanation : Let $y = x^x$

$\Rightarrow \quad \log y = x\log x$

Differentiating w.r.t. x

$$\Rightarrow \quad \frac{1}{y}\frac{dy}{dx} = x\left(\frac{1}{x}\right)+\log x(1)$$
$$\frac{dy}{dx} = y(1+\log x)$$
$$= x^x(1+\log x)$$

Hence R is true.

Since $\frac{d}{dx}(x^x) = x^x(1+\log x)$

$$\int x^x(1+\log x)dx = x^x + c$$

Using the concept of anti-derivative, A is true. R is the correct explanation for A.

Q. 4. Assertion (A): $\int x^2 dx = \frac{x^3}{3}+c$

Reason (R): $\int e^{x^2}dx = e^{x^{3}/3}+c$

Ans. Option (C) is correct.

Explanation:

Since $\int x^n\,dx = \frac{x^{n+1}}{n+1}+c,$

$$\int x^2 dx = \frac{x^{2+1}}{2+1}+c$$
$$= \frac{x^3}{3}+c$$

$\therefore$ *A* is true.

$\int e^{x^2}dx$ is a function which can not be integrated.

$\therefore$ R is false.

Q. 5. Assertion (A): $\int_0^{\pi/2} \frac{\cos x}{\sin x + \cos x}dx = \frac{\pi}{4}$

Reason (R): $\int_0^{\pi/2} \frac{\sin x}{\sin x + \cos x}dx = \frac{\pi}{4}$

Ans. Option (A) is correct.

Explanation:

Let $$I = \int_0^{\pi/2} \frac{\sin x}{\sin x + \cos x}\,dx \quad ...(i)$$

$$\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$$

$$\therefore \quad I = \int_0^{\pi/2} \frac{\sin\left(\frac{\pi}{2}-x\right)dx}{\sin\left(\frac{\pi}{2}-x\right)+\cos\left(\frac{\pi}{2}-x\right)}$$

$$I = \int \frac{\cos x}{\cos x + \sin x}\,dx \quad ...(ii)$$

Adding equations (i) + (ii),

$$\Rightarrow \quad 2I = \int_0^{\pi/2} \frac{\sin x + \cos x}{\sin x + \cos x}\,dx$$

$$= \int_0^{\pi/2} 1\,dx$$

$$= [x]_0^{\pi/2}$$

$$= \frac{\pi}{2}$$

$$\therefore \quad I = \frac{\pi}{4}$$

Hence R is true.
From (ii), A is also true.
R is the correct explanation for A.

Q. 6. Assertion (A): $\int_{-3}^{3} (x^3+5)dx = 30$

Reason (R): $f(x) = x^3 + 5$ is an odd function.

Ans. Option (C) is correct.

Explanation:

Let $f(x) = x^3 + 5$

$f(-x) = (-x)^3 + 5$

$= -x^3 + 5$

$f(x)$ is neither even nor odd. Hence R is false.

$$\int_{-3}^{3} x^3\,dx = 0 \qquad [\because x^3 \text{ is odd}]$$

$$\int_{-3}^{3} 5\,dx = 5[x]_{-3}^{3} = 30$$

$$\therefore \int_{-3}^{3} (x^3+5)dx = 0 + 30 = 30$$

Hence A is true.

Q. 7. Assertion (A): $\frac{d}{dx}\left[\int_0^{x^2} \frac{dt}{t^2+4}\right] = \frac{2x}{x^4+4}$

Reason (R): $\int \frac{dx}{x^2+a^2} = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + c$

Ans. Option (A) is correct.

Explanation:

$$\int \frac{dx}{x^2+a^2} = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + c.$$

This is a standard integral and hence true.
So R is true.

$$\int_0^{x^2} \frac{dt}{t^2+4} = \left[\frac{1}{2}\tan^{-1}\left(\frac{t}{2}\right)\right]_0^{x^2}$$

$$= \frac{1}{2}\tan^{-1}\left(\frac{x^2}{2}\right)$$

$$\frac{d}{dx}\left[\int_0^{x^2} \frac{dt}{t^2+4}\right] = \frac{d}{dx}\left[\frac{1}{2}\tan^{-1}\left(\frac{x^2}{2}\right)\right]$$

$$= \frac{1}{2} \times \frac{1}{1+\frac{x^4}{4}} \times \frac{2x}{2}$$

$$= \frac{x}{2} \times \frac{4}{4+x^4}$$

$$= \frac{2x}{4+x^4}$$

Hence A is true and R is the correct explanation for A.

Q. 8. Assertion (A): $\int_{-1}^{1} (x^3 + \sin x + 2)dx = 0$

Reason (R):

$$\int_{-a}^{a} f(x)dx = \begin{cases} 2\int_0^a f(x)dx, & \text{if } f(x) \text{ is an even function i.e.,} (-x) = f(x) \\ 0, & \text{if } f(x) \text{ is an odd function i.e., } f(-x) = -f(x) \end{cases}$$

Ans. Option (D) is correct.

Explanation:

$$\int_{-a}^{a} f(x)dx = \begin{cases} 2\int_0^a f(x)dx, & \text{if } f(x) \text{ is an even function i.e.,} (-x) = f(x) \\ 0, & \text{if } f(x) \text{ is an odd function i.e., } f(-x) = -f(x) \end{cases}$$

This is a property of the definite integrals and hence R is true.

$$\int_{-1}^{1} (x^3 + \sin x + 2)dx$$

$$= \underbrace{\int_{-1}^{1} (x^3 + \sin x)dx}_{\text{Odd function}} + \underbrace{\int_{-1}^{1} 2\,dx}_{\text{Even function}}$$

$$= 0 + 2[x]_{-1}^{1}$$

$$= 2 \times 2$$

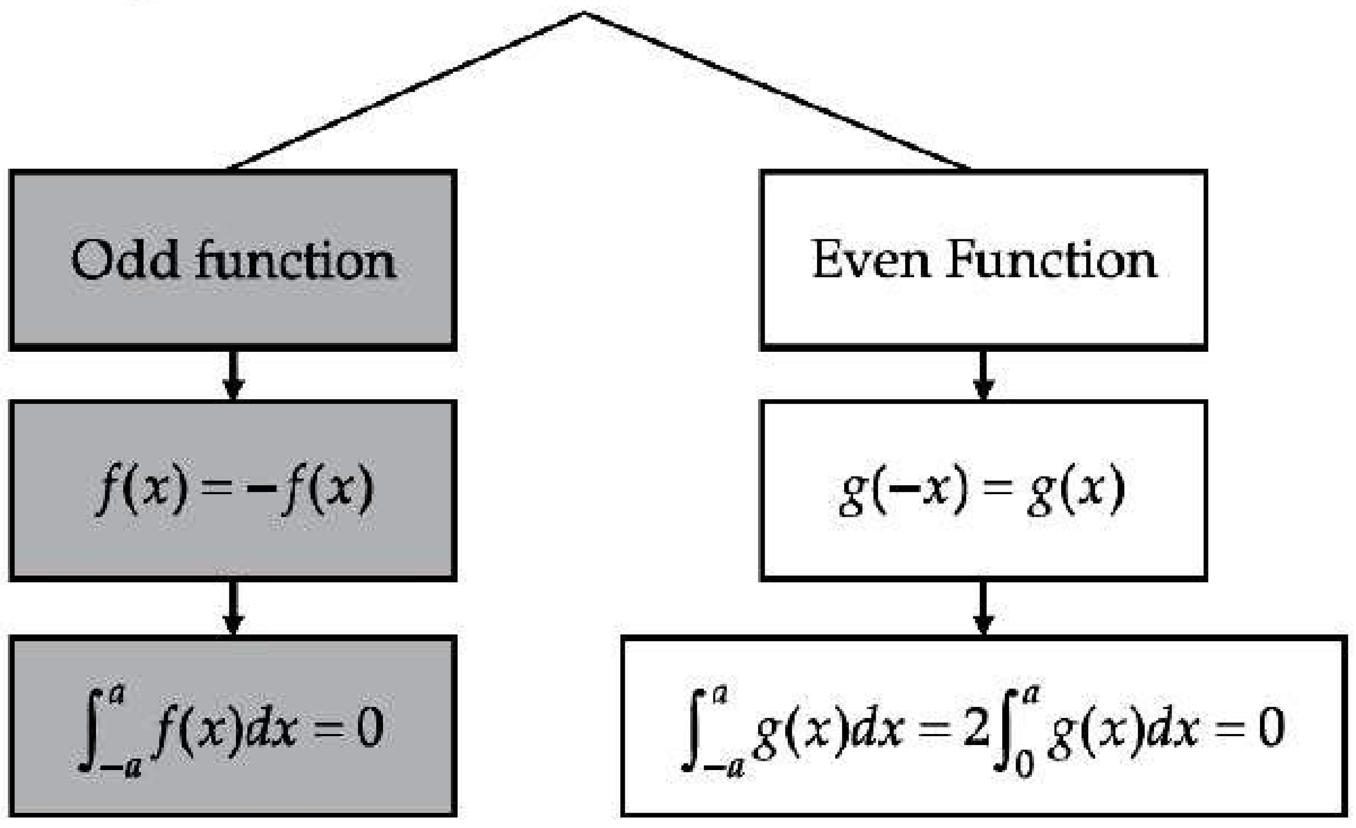
$$= 4$$

Hence A is false.

CASE-BASED MCQs

Attempt any four sub-parts from each question. Each sub-part carries 1 mark.

I. Read the following text and answer the following questions on the basis of the same:



Q. 1. $\int_{-1}^{1} x^{99} dx =$ ______.

(A) 0 (B) 1
(C) –1 (D) 2

Ans. Option (A) is correct.

Explanation:

$\int_{-1}^{1} x^{99} dx = 0$, since x^{99} is an odd function.

Q. 2. $\int_{-\pi}^{\pi} x\cos x\, dx =$ ______.

(A) 1 (B) 0
(C) –1 (D) $\frac{\pi}{2}$

Ans. Option (B) is correct.

Explanation:

$\int_{-\pi}^{\pi} x\cos x\, dx = 0$, since $x\cos x$ is an odd function.

Q. 3. $\int_{-\pi/2}^{\pi/2} \sin^3 x\, dx =$ ______.

(A) 1 (B) 0
(C) –1 (D) π

Ans. Option (B) is correct.

Explanation:

$\int_{-\pi/2}^{\pi/2} \sin^3 x\, dx = 0$, since $\sin^3 x$ is an odd function.

Q. 4. $\int_{-\pi}^{\pi} x\sin x\, dx =$ ______.

(A) π (B) 0
(C) 2π (D) $\frac{\pi}{2}$

Ans. Option (C) is correct.

Explanation: Since, $x\sin x$ is an even function

$$\begin{aligned}\int_{-\pi}^{\pi} x\sin x\, dx &= 2\int_0^{\pi} x\sin x\, dx \\ &= 2\left[-x\cos x + \int (1\times\cos x\, dx)\right]_0^{\pi} \\ &= 2\left[-x\cos x + \sin x\right]_0^{\pi} \\ &= 2[(\pi+0)-(0+0)] \\ &= 2\pi\end{aligned}$$

Q. 5. $\int_{-\pi}^{\pi} \tan x \sec^2 x\, dx =$ ______.

(A) 1 (B) –1
(C) 0 (D) 2

Ans. Option (C) is correct.

Explanation:

$\int_{-\pi}^{\pi} \tan x \sec^2 x\, dx = 0$, Since it is an odd function

II. Read the following text and answer the following questions on the basis of the same:

$$\begin{aligned}\int e^x[f(x)+f'(x)]dx &= \int e^x f(x)dx + \int e^x f'(x)dx \\ &= f(x)e^x - \int f'(x)e^x dx \\ &\quad + \int f'(x)e^x dx \\ &= e^x f(x) + c\end{aligned}$$

Q. 1. $\int e^x(\sin x + \cos x)dx =$ ______.

(A) $e^x \cos x + c$ (B) $e^x \sin x + c$
(C) $e^x + c$ (D) $e^x(-\cos x + \sin x) + c$

Ans. Option (B) is correct.

Explanation:

$$\int e^x(\underbrace{\sin x}_{f(x)} + \underbrace{\cos x}_{f'(x)})dx = e^x \sin x + c$$

Q. 2. $\int e^x\left(\frac{x-1}{x^2}\right)dx =$ ______.

(A) $e^x + c$ (B) $\frac{e^x}{x} + c$
(C) $\frac{e^x}{x^2} + c$ (D) $\frac{-e^x}{x^2} + c$

Ans. Option (B) is correct.

Explanation:

$$\begin{aligned}\int e^x\left(\frac{x-1}{x^2}\right)dx &= \int e^x\left(\underbrace{\frac{1}{x}}_{f(x)} - \underbrace{\frac{1}{x^2}}_{f'(x)}\right)dx \\ &= \frac{e^x}{x} + c\end{aligned}$$

Q. 3. $\int e^x(x+1)dx =$ ______.

(A) $xe^x + c$ (B) $e^x + c$

(C) $e^{-x} + c$ (D) None of these

Ans. Option (A) is correct.

Explanation:

$$\int e^x(\underbrace{x}_{f(x)} + \underbrace{1}_{f'(x)})dx = xe^x + c$$

Q. 4. $\int_0^{\pi} e^x(\tan x + \sec^2 x)dx =$ ______.

(A) 0 (B) 1

(C) –1 (D) $-e^{\pi}$

Ans. Option (A) is correct.

Explanation:

$$\int_0^{\pi} e^x(\tan x + \sec^2 x)dx = \left[e^x \tan x\right]_0^{\pi} = 0$$

Q. 5. $\int \frac{xe^x}{(1+x)^2}dx =$ ______.

(A) $x e^x + c$ (B) $\frac{e^x}{(x+1)^2}+c$

(C) $\frac{xe^x}{x+1}+c$ (D) $\frac{e^x}{x+1}+c$

Ans. Option (D) is correct.

Explanation:

$$\int e^x\left[\frac{(x+1)-1}{(x+1)^2}\right]dx = \int e^x\left[\frac{1}{x+1}-\frac{1}{(x+1)^2}\right]dx$$
$$= \frac{e^x}{x+1}+c$$

III. Read the following text and answer the following questions on the basis of the same:

Let's say that we want to evaluate $\int[P(x)/Q(x)]\,dx$, where $P(x)/Q(x)$ is a proper rational fraction. In such cases, it is possible to write the integrand as a sum of simpler rational functions by using partial fraction decomposition. Post this, integration can be carried out easily. The following image indicates some simple partial fractions which can be associated with various rational functions:

S. No.	Form of the rational function	Form of the partial fraction
1.	$\frac{px+q}{(x-a)(x-b)}, a \neq b$	$\frac{A}{x-a}+\frac{B}{x-b}$
2.	$\frac{px+q}{(x-a)^2}$	$\frac{A}{x-a}+\frac{B}{(x-a)^2}$
3.	$\frac{px^2+qx+r}{(x-a)(x-b(x-c)}$	$\frac{A}{x-a}+\frac{B}{x-b}+\frac{C}{x-c}$
4.	$\frac{px^2+qx+r}{(x-a)^2(x-b)}$	$\frac{A}{x-a}+\frac{B}{(x-a)^2}+\frac{C}{x-b}$
5.	$\frac{px^2+qx+r}{(x-a)(x^2+bx+c)}$	$\frac{A}{x-a}+\frac{Bx+C}{x^2+bx+c}$,
	where, $x^2 + bx + c$ cannot be factorised further	

In the above table, A, B and C are real numbers to be determined suitably.

Q. 1. $\int \frac{dx}{(x+1)(x+2)}$

(A) $\log\left|\frac{x+1}{x+2}\right|+C$ (B) $\log\left|\frac{x-1}{x+2}\right|+C$

(C) $\log\left|\frac{x+1}{x-2}\right|+C$ (D) $\log\left|\frac{x+2}{x+1}\right|+C$

Ans. Option (A) is correct.

Explanation: We write,

$$\frac{1}{(x+1)(x+2)} = \frac{A}{x+1}+\frac{B}{x+2} \quad \text{...(i)}$$

where, real number A and B are to be determined suitably. This gives

$$1 = A(x+2)+B(x+1)$$

Equating the coefficients of x and the constant term, we get

$$A+B = 0$$

and $2A+B = 1$

Solving these equations, we get $A = 1$ and $B = -1$. Thus, the integrand is given by

$$\frac{1}{(x+1)(x+2)} = \frac{1}{x+1}+\frac{-1}{x+2}$$

Therefore,

$$\int\frac{dx}{(x+1)(x+2)} = \int\frac{dx}{x+1}-\int\frac{dx}{x+2}$$
$$= \log|x+1|-\log|x+2|+C$$
$$= \log\left|\frac{x+1}{x+2}\right|+C$$

Q. 2. Integration of $\frac{x}{(x+1)(x+2)}$

(A) $\log\frac{(x+1)^2}{(x+2)}+C$ (B) $\log\frac{(x+2)^2}{(x+1)}+C$

(C) $\log\frac{(x)^2}{(x+1)}+C$ (D) $\log\frac{(x-2)^2}{(x+1)}+C$

Ans. Option (B) is correct.

Explanation:

Let $\frac{x}{(x+1)(x+2)} = \frac{A}{(x+1)}+\frac{B}{(x+2)}$

$\Rightarrow$ $x = A(x+2)+B(x+1)$

Equating the coefficients of x and constant term, we obtain

$$A+B = 1$$
$$2A+B = 0$$

On solving, we obtain

$$A = -1 \text{ and } B=2$$

$$\therefore \quad \frac{x}{(x+1)(x+2)} = \frac{-1}{(x+1)}+\frac{2}{(x+2)}$$

$$\Rightarrow \int \frac{x}{(x+1)(x+2)}dx = \int \frac{-1}{(x+1)}+\frac{2}{(x+2)}dx$$
$$= -\log|x+1|+2\log|x+2| +C$$
$$= \log(x+2)^2-\log|x+1|+C$$
$$= \log\frac{(x+2)^2}{(x+1)}+C$$

Q. 3. $\int \frac{1}{x^2-9}dx$

(A) $\frac{1}{6}\log\left|\frac{x-3}{x+3}\right|+C$ (B) $\frac{1}{6}\log\left|\frac{x-2}{x+3}\right|+C$

(C) $\frac{1}{6}\log\left|\frac{x+3}{x-3}\right|+C$ (D) $\frac{1}{3}\log\left|\frac{x-3}{x+3}\right|+C$

Ans. Option (A) is correct.

Explanation:

$$\text{Let} \quad \frac{1}{(x+3)(x-3)} = \frac{A}{(x+3)}+\frac{B}{(x-3)}$$
$$1 = A(x-3)+B(x+3)$$

Equating the coefficients of x and constant term, we obtain

$$A+B = 0$$
$$-3A+3B = 1$$

On solving, we obtain

$$A = -\frac{1}{6} \text{ and } B=\frac{1}{6}$$

$$\therefore \quad \frac{1}{(x+3)(x-3)} = \frac{-1}{6(x+3)}+\frac{1}{6(x-3)}$$

$$\Rightarrow \int\frac{1}{(x^2-9)}dx = \int\left(\frac{-1}{6(x+3)}+\frac{1}{6(x-3)}\right)dx$$
$$= -\frac{1}{6}\log|x+3|+\frac{1}{6}\log|x-3| +C$$
$$= \frac{1}{6}\log\left|\frac{(x-3)}{(x+3)}\right|+C$$

Q. 4. $\int\frac{1}{e^x-1}dx =$

(A) $\log\left|\frac{e^x-1}{2}\right|+C$ (B) $\log\left|\frac{e^x-1}{2e^x}\right|+C$

(C) $\log\left|\frac{e^x-1}{2x}\right|+C$ (D) $\log\left|\frac{e^x-1}{e^x}\right|+C$

Ans. Option (D) is correct.

Explanation:

$$\text{Let} \quad e^x = t$$
$$\Rightarrow \quad e^x\,dx = dt$$
$$\Rightarrow \quad \int\frac{1}{e^x-1}dx = \int\frac{1}{t-1}\times\frac{dt}{t}$$
$$= \int\frac{1}{t(t-1)}dt$$

$$\text{Let} \quad \frac{1}{t(t-1)} = \frac{A}{t}+\frac{B}{t-1}$$
$$1 = A(t-1)+Bt \qquad \text{...(i)}$$

Substituting $t = 1$ and $t = 0$ in equation (i), we obtain

$A = -1$ and $B = 1$

$$\therefore \quad \frac{1}{t(t-1)} = \frac{-1}{t}+\frac{1}{t-1}$$

$$\Rightarrow \quad \int\frac{1}{t(t-1)}dt = \log\left|\frac{t-1}{t}\right|+C$$
$$= \log\left|\frac{e^x-1}{e^x}\right|+C$$

Q. 5. $\int\frac{dx}{x(x^2+1)} =$

(A) $\log|x|+\frac{1}{2}\log|x^2+1|+C$

(B) $\log|x|-\frac{1}{4}\log|x^2+1|+C$

(C) $\log|x|-\frac{1}{2}\log|x^2+1|+C$

(D) $\log|x|-\frac{1}{3}\log|x^2-1|+C$

Ans. Option (C) is correct.

Explanation:

$$\text{Let} \quad \frac{1}{x(x^2+1)} = \frac{A}{x}+\frac{Bx+C}{x^2+1}$$
$$1 = A(x^2+1)+(Bx+C)x$$

Equating the coefficients of x^2, x and constant term, we obtain

$$A+B = 0$$
$$C = 0$$
$$A = 1$$

On solving these equations, we obtain

$A=1, B=-1,$ and $C=0$

$$\therefore \quad \frac{1}{x(x^2+1)} = \frac{1}{x}+\frac{-x}{x^2+1}$$

$$\Rightarrow \quad \int\frac{1}{x(x^2+1)}dx = \int\left\{\frac{1}{x}-\frac{x}{x^2+1}\right\}dx$$
$$= \log|x|-\frac{1}{2}\log|x^2+1|+C$$